

THE INDEX OF RUBIN-STARK UNITS

SAAD EL BOUKHARI⁽¹⁾ & YOUNESS MAZIGH^{(1),(2)}

ABSTRACT. We compare the p -parts of the orders of the class groups and the quotients of S -units by the Rubin-Stark units along the cyclotomic tower over a totally real field.

1. INTRODUCTION AND PRELIMINARES

The class number associated with a number field is known to be related to L -functions, and this can provide valuable information about class groups using computations of special values of those functions.

A direct way to link those two concepts is based on what is called class number formulas. Those formulas can be described as well by Iwasawa theory at "infinite level" in what is called "main conjectures". A localisation at a certain prime p is, often, technically better than the global approach. This strategy of "prime" by "prime" problem solving requires the introduction of the concept of p -adic L -functions which are the p -adic interpolations of L -functions and agree with them at negative integers.

In many cases, there is not enough information concerning the existence nor the exact properties of such interpolations. Luckily enough, the work of Iwasawa and other authors shows that p -adic L -functions are in many known cases equivalent to quotients of a group of units by a subgroup of special units. This is the case, for example, of circular units in the abelian setting. Iwasawa has proved that the "plus" part of the quotient of local units by circular units is mainly pseudo-isomorphic to the quotient of Iwasawa's algebra by the series associated with the p -adic L function. This result is also described in the famous class number formula that links the index of circular units within the group of units and the class number in the totally real abelian case.

Class number formulas where the class number is compared to the index of special units within their group of units have been formulated in the abelian and imaginary cases for circular and elliptic units respectively. It seems, however, that such results that would use the Rubin-Stark units are absent from literature and it is in this perspective that this work has been conducted.

This paper has therefore for aim to formulate and prove a class number formula which involves the index of Rubin-Stark units within the group of S -units. We introduce first some of the notations that will be used for this purpose.

Let k be a totally real field of degree $r = [k : \mathbb{Q}]$ and let K/k be a finite abelian extension of totally real number fields with Galois group G . Fix a finite set S of places of k containing all infinite places and all places ramified in K/k , and a second finite set T of places of k , disjoint from S . Let $\hat{G} = \text{Hom}(G, \mathbb{C}^\times)$. If $\chi \in \hat{G}$ we define the modified Artin L -function attached to χ by

$$L_{S,T}(s, \chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\sigma_{\mathfrak{p}}) \mathbf{N}\mathfrak{p}^{-s})^{-1} \prod_{\mathfrak{p} \in T} (1 - \chi(\sigma_{\mathfrak{p}}) \mathbf{N}\mathfrak{p}^{1-s})$$

where $\sigma_{\mathfrak{p}} \in G$ is the Frobenius of the (unramified) prime \mathfrak{p} .

2010 *Mathematics Subject Classification.* 11R23, 11R27, 11R29.

Key words and phrases. Iwasawa theory, Stark's conjecture, L -functions.

For each $\chi \in \widehat{G}$, there is an idempotent

$$e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1} \in \mathbb{C}[G].$$

Following [9] we define the Stickelberger element

$$\Theta_{S,T}(s) = \Theta_{S,T,K/k}(s) = \sum_{\chi \in \widehat{G}} L_{S,T}(s, \chi^{-1}) e_\chi$$

which we view as a $\mathbb{C}[G]$ -valued meromorphic function on \mathbb{C} . Let $\chi \in \widehat{G}$ and let $r_S(\chi)$ be the order of vanishing of $L_{S,T}(s, \chi)$ at $s = 0$. Recall that

$$r_S(\chi) = \text{ord}_{s=0} L_{S,T}(s, \chi) = \begin{cases} |\{v \in S : \chi(D_v(K/k)) = 1\}|, & \chi \neq 1; \\ |S| - 1, & \chi = 1. \end{cases}$$

(see for example [9] Proposition I.3.4), where $D_v(K/k)$ is the decomposition group of v relative to K/k .

Before stating the Rubin-Stark conjecture we record some hypotheses $\mathbf{H}(F/K, S, \mathcal{T}, r)$:

- (1) S contains all the infinite primes of k and all the primes which ramify in K/k ;
- (2) S contains at least r places which split completely in K/k ;
- (3) $|S| \geq r + 1$;
- (4) $T \neq \emptyset$, $S \cap T = \emptyset$ and $U_{S,T}(K)$ is torsion-free,

Here $U_{S,T}(K)$ is the group of S -units of K which are congruent to 1 modulo all the primes in T .

Conditions (2) and (3) ensure that $s^{-r} \Theta_{S,T}(s)$ is holomorphic at $s = 0$. Since K/k is an extension of totally real fields and S contains all infinite places the second condition is satisfied by default. The condition (4) is easily satisfied. For example, if T contains primes of two different residue characteristics.

For any set V of places of k , we denote by V_K the set of places of K lying above places in V . Let M be a \mathbb{Z} -module. If \mathbf{R} is one of the fields \mathbb{Q}, \mathbb{R} or \mathbb{C} , we note by $\mathbf{R}M$ the tensorial product $\mathbf{R} \otimes_{\mathbb{Z}} M$.

We write $S = S_\infty \cup V$ so that $\mathbb{R}S_K = \mathbb{R}S_{\infty,K} \oplus \mathbb{R}V_K$ (as $\mathbb{R}[G]$ -modules) and let π_∞ denote the projection from $\mathbb{R}S_K$ to $\mathbb{R}S_{\infty,K}$. We define $\mathcal{L}_{S,\infty}$ as the composite $\pi_\infty \circ \mathcal{L}_S$:

$$\mathcal{L}_{S,\infty} : \mathbb{R}U_{S,T}(K) \xrightarrow{\mathcal{L}_S} \mathbb{R}S_K \xrightarrow{\pi_\infty} \mathbb{R}S_{\infty,K} \quad (1)$$

where \mathcal{L}_S is a logarithmic 'embedding' of $U_{S,T}(K)$:

$$\begin{aligned} \mathcal{L}_S : U_{S,T}(K) &\rightarrow \mathbb{R}S_K := \bigoplus_{w \in S_K} \mathbb{R}w \\ \varepsilon &\mapsto -\sum_{w \in S_K} \log(|\varepsilon|_w) w. \end{aligned}$$

Taking r -th exterior powers over the commutative ring $\mathbb{R}[G]$ gives an $\mathbb{R}[G]$ -linear map

$$\bigwedge_{\mathbb{R}[G]}^r \mathcal{L}_{S,\infty} : \bigwedge_{\mathbb{R}[G]}^r U_{S,T}(K) \longrightarrow \bigwedge_{\mathbb{R}[G]}^r \mathbb{R}S_{\infty,K} = \mathbb{R}[G](w_1 \wedge \dots \wedge w_r)$$

where w_1, \dots, w_r is a choice of r -places of K above the infinite places $\{v_1, \dots, v_r\}$ of k . Since $w_1 \wedge \dots \wedge w_r$ is a free generator we can define a unique $\mathbb{R}[G]$ -linear 'regulator' R_w , called Rubin-Stark regulator:

$$\bigwedge_{\mathbb{R}[G]}^r U_{S,T}(K) \longrightarrow \mathbb{R}[G] \text{ by } \bigwedge_{\mathbb{R}[G]}^r \mathcal{L}_{S,\infty}(x) = R_w(x)(w_1 \wedge \dots \wedge w_r).$$

Explicitly, every element of $\mathbb{R} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)$ is a finite sum of terms of form $\varepsilon_1 \wedge \cdots \wedge \varepsilon_r$ with $\varepsilon_i \in \mathbb{R} U_{S,T}(K)$ and

$$\begin{aligned} R_w : \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K) &\longrightarrow \mathbb{R}[G] \\ \varepsilon = \varepsilon_1 \wedge \cdots \wedge \varepsilon_r &\longmapsto \det(-\sum_{\sigma \in G} \log |\varepsilon_i^\sigma|_{w_j} \sigma^{-1})_{i,j=1}^r \end{aligned}$$

Let $\Theta_{S,T}^{(r)}(0)$ be the coefficient of s^r in the Taylor series of $\Theta_{S,T}$;

$$\Theta_{S,T}^{(r)}(0) := \lim_{s \rightarrow 0} s^{-r} \Theta_{S,T}^{(r)}(s).$$

Conjecture B' (Rubin-Stark conjecture) of [8] predicts the existence of certain elements

$$\eta_{K,S,T} \in \Lambda_{S,T} \subset \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K) \text{ such that } R_w(\eta_{K,S,T}) = \Theta_{S,T}^{(r)}(0)$$

where the module $\Lambda_{S,T}$ is defined in [8, §2.1]. We assume the validity of this conjecture and define Stark_K to be the module generated as a $\mathbb{Z}[G]$ -module by the following Rubin-Stark elements:

$$\{\eta_F \in \mathbb{Q} \bigwedge_{\mathbb{Z}[\text{Gal}(F/k)]}^r U_{S,T}(F), k \subseteq F \subseteq K\}$$

For any $\mathbb{Z}[G]$ -module A , we note

$$e_{S,r}A := e_{S,r}(A \otimes \mathbb{Q})$$

where $e_{S,r} = \sum_{\chi \in \hat{G}, r_S(\chi)=r} e_\chi$. And $A^{[S,r]}$ refers to a choice of a lattice inside the \mathbb{Q} -vector space $e_{S,r}A$. In this paper we prove

Theorem 1.1. *The index $[\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}]$ is finite, and we have*

$$[\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}] = [\mathbb{Z}[G]^{[S,r]} : U(K)^{[S,r]}] h_{K,S,T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} h_{K_I, S, T}^{(-1)^{|I|}}$$

where $U(K)$ is the Sinnott module (see Definition 3.2).

Let $K_\infty = \bigcup_{n \geq 0} K_n$ be the cyclotomic \mathbb{Z}_p -extension of K and let G_n denote the Galois group $\text{Gal}(K_n/k)$. It would be interesting to be able to control the finite indices $[\bigwedge_{\mathbb{Z}[G_n]}^r U_{S,T}(K_n)^{[S,r]} : \text{Stark}_{K_n}^{[S,r]}]$ along the cyclotomic tower.

For this purpose, we prove the following result

Theorem 1.2. *There exists factors B_n bounded asymptotically for n sufficiently large such that*

$$[\bigwedge_{\mathbb{Z}[G_n]}^r U_{S,T}(K_n)^{[S,r]} : \text{Stark}_{K_n}^{[S,r]}] = h_{K_n, S, T} B_n,$$

where $h_{K_n, S, T}$ is the S -class number of K_n modulo T .

The index formula presented in this paper will be used in an up coming work to prove a main conjecture based on a divisibility result shown in [3]. The techniques used in [1] can then be applied to prove ETNC.

2. IMAGE BY THE RUBIN-STARK REGULATOR

Throughout this section, let F/k be an intermediate extension in K/k , η_F a Rubin-Stark unit in F . Let H (resp. Δ) denote the Galois group $\text{Gal}(K/F)$ (resp. $\text{Gal}(F/k)$).

Proposition 2.1. *Let $R_{w'}$ be the restriction of the regulator map R_w to the subfield F defined by using the infinites places w'_1, \dots, w'_r of F below the places w_1, \dots, w_r of K . then*

$$R_w(\eta_F) = |H|^{2r} R_{w'}(\eta_F)$$

Proof. Let η_F be a Rubin-Stark unit in F . By definition

$$R_w(\eta_F) = \det_{i,j}(a_{i,j})$$

where

$$a_{i,j} = -\sum_{\sigma \in G} \log |(\eta_F)_i^{\sigma^{-1}}|_{w_j} \sigma$$

here we note $\eta_F = (\eta_F)_1 \wedge \dots \wedge (\eta_F)_i \wedge \dots \wedge (\eta_F)_r$. Let us first calculate the coefficient $a_{i,j}$ for some given (i,j) . To simplify notations we refer to $(\eta_F)_i$ simply as η . Then

$$\begin{aligned} \sum_{\sigma \in G} \log | \eta^{\sigma^{-1}} |_{w_j} \sigma &= \sum_{\delta \in \Delta} \sum_{h \in H} \log | \eta^{\delta^{-1}h^{-1}} |_{w_j} \delta h \\ &= \sum_{\delta \in \Delta} \sum_{h \in H} \log | \eta^{\delta^{-1}} |_{w_j} \delta h \\ &= \sum_{h \in H} \sum_{\delta h^{-1} \in \Delta} \log | \eta^{\delta^{-1}} |_{w_j} \delta \\ &= | H | \sum_{\delta \in \Delta} \log | \eta^{\delta^{-1}} |_{w_j} \delta \\ &= | H | \sum_{\delta \in \Delta} \log | N_{K/F}(\eta)^{\delta^{-1}} |_{w_{j'}} \delta, \quad w_{j'} := w_j \cap F \\ &= | H |^2 \sum_{\delta \in \Delta} \log | \eta^{\delta^{-1}} |_{w_{j'}} \delta. \end{aligned}$$

Finally we have

$$R_w(\eta_F) = | H |^{2r} R_{w'}(\eta_F)$$

where $R_{w'}$ is the same as R_w but defined over F instead of K using the infinite places w'_1, \dots, w'_r of F below the places w_1, \dots, w_r of K . \square

For a character $\chi \in \hat{\Delta}$, let F^χ denote the fixed field of $\ker(\chi)$, \mathfrak{f}_χ the conductor of F^χ/k and \mathfrak{f}_F the conductor of F/k . Let \mathfrak{p} be a prime of k , $I_{\mathfrak{p}}(F/k)$ the inertia group of \mathfrak{p} relative to F/k and let

$$e_{I_{\mathfrak{p}}} = \frac{\sum_{\sigma \in I_{\mathfrak{p}}(F/k)} \sigma}{| I_{\mathfrak{p}}(F/k) |}.$$

Then we have the following proposition

Proposition 2.2. *There exists an element $\omega_K \in \mathbb{R}[G]$ independent of the choice of the field F which verifies*

$$R_{w'}(\eta_F) = \omega_K \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} | \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right).$$

Proof. As we previously stated

$$R_{w'}(\eta_F) = \Theta_{S,T}^{(r)}(0) = \sum_{\chi \in \hat{\Delta}} L_{S,T}^{(r)}(0, \chi) e_{\chi^{-1}}.$$

For any character χ of $\hat{\Delta}$, let $\hat{\chi}$ denote the associated primitive character obtained by restricting χ to $\Delta/\ker(\chi)$ (so that we obtain a faithful character). We have then the following equalities

$$\begin{cases} L_S^{(r)}(0, \chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\sigma_{\mathfrak{p}}))^{-1} \\ L_S^{(r)}(0, \hat{\chi}) = \prod_{\mathfrak{p} \notin S} (1 - \hat{\chi}(\sigma_{\mathfrak{p}}))^{-1} \end{cases}$$

here $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism associated with the prime \mathfrak{p} . If $\mathfrak{p} \notin S$ then it is unramified in F^χ/k and therefore

$$\chi(\sigma_{\mathfrak{p}}) = \hat{\chi}(\sigma_{\mathfrak{p}}).$$

If \mathfrak{p} is ramified in F/k but unramified in F^χ/k then $\chi(\sigma_{\mathfrak{p}}) = 0$ and $\hat{\chi}(\sigma_{\mathfrak{p}}) \neq 0$, and if not $\chi(\sigma_{\mathfrak{p}}) = 0 = \hat{\chi}(\sigma_{\mathfrak{p}})$. Consequently,

$$L_S^{(r)}(0, \chi) = \left(\prod_{\mathfrak{p} | \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \hat{\chi}(\sigma_{\mathfrak{p}})) \right) L_S^{(r)}(0, \hat{\chi}).$$

Hence

$$\begin{aligned} R_{w'}(\eta_F) &= \sum_{\chi \in \hat{\Delta}} L_{S,T}^{(r)}(0, \chi) e_{\chi^{-1}} \\ &= \sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \left(\prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \hat{\chi}(\sigma_{\mathfrak{p}})) \right) L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}}. \end{aligned}$$

Since $\sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}} e_{\chi^{-1}} = \hat{\chi}(\sigma_{\mathfrak{p}}) e_{\chi^{-1}}$ holds, we obtain

$$\begin{aligned} R_{w'}(\eta_F) &= \sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \left(\prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \hat{\chi}(\sigma_{\mathfrak{p}})) \right) L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}} \\ &= \sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \left(\prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) (L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}}) \\ &= \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}} \right). \end{aligned}$$

Each character of $\Delta = \text{Gal}(F/k)$ can be seen as a character of $G = \text{Gal}(K/k)$ trivial on $H = \text{Gal}(K/F)$. An idempotent of such a character annihilates, by multiplication, any other idempotent associated with a character of G that happens to be not trivial on H . This is why we have

$$\begin{aligned} & \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}} \right) \\ &= \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \left(\sum_{\chi \in \hat{G}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}} \right), \end{aligned}$$

which gives if we note

$$\omega_K := \sum_{\chi \in \hat{G}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}}$$

the following

$$R_{w'}(\eta_F) = \omega_K \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right).$$

□

We combine the results of the two previous sections and get

Proposition 2.3. *Recall that $H := \text{Gal}(K/F)$ and also the notation $e_{S,r} = \sum_{\chi \in \hat{G}, r_S(\chi)=r} e_{\chi}$. Then*

$$R_w(\eta_F) = \omega_K \left(\left(\sum_{\sigma \in H} \sigma \right)^{2r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) \right) e_{S,r}$$

Proof. Propositions 2.1 and 2.2 give

$$\begin{aligned} R_w(\eta_F) &= |H|^{2r} R_{w'}(\eta_F) \\ &= |H|^{2r} \omega_K \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F, \mathfrak{p} \nmid \mathfrak{f}_\chi} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \end{aligned}$$

In the last equality we can eliminate the condition $\mathfrak{p} \nmid \mathfrak{f}_\chi$ from the product since in the case when $\mathfrak{p} \mid \mathfrak{f}_\chi$, $\hat{\chi}(\sigma_{\mathfrak{p}}) = 0$. Then we have

$$\begin{aligned} R_w(\eta_F) &= |H|^{2r} \omega_K \left(\sum_{\chi \in \hat{\Delta}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \\ &= \omega_K \left(\left(\sum_{\sigma \in H} \sigma \right)^{2r} \sum_{\chi \in \hat{G}, r_S(\chi)=r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} \right) \end{aligned}$$

To make sense of the last equality note that

$$\begin{cases} \text{If } \chi \in \hat{G} \text{ and } \chi(H) \neq 1 \text{ then } \left(\sum_{\sigma \in H} \sigma \right) e_{\chi^{-1}} = 0 \\ \text{If } \chi \in \hat{G} \text{ and } \chi(H) = 1 \text{ then } \left(\sum_{\sigma \in H} \sigma \right) e_{\chi^{-1}} = |H| e_{\chi^{-1}} \end{cases}$$

We conclude that

$$R_w(\eta_F) = \omega_K \left(\left(\sum_{\sigma \in H} \sigma \right)^{2r} \prod_{\mathfrak{p} \mid \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) \right) e_{S,r}.$$

□

3. INDEX OF THE "STARK" MODULE

3.1. The generalised Sinnott index. We recall some data about the generalised Sinnott index. For a more complete exhibit of the properties of this index the reader is invited to refer to [7]. Let p be a prime rational and v_p its normalised valuation ($v_p(p) = 1$). Let \mathbb{F} be one of the fields \mathbb{Q} , \mathbb{Q}_p or \mathbb{R} , and let

$$\mathcal{O} := \begin{cases} \mathbb{Z}, & \mathbb{F} = \mathbb{Q} \text{ or } \mathbb{R}; \\ \mathbb{Z}_p, & \mathbb{F} = \mathbb{Q}_p. \end{cases}$$

Let E be an \mathbb{F} -vector space of finite dimension d . An \mathcal{O} -lattice Λ is a free \mathcal{O} -submodule of E of rank d such that the \mathbb{F} -vector space generated by Λ is E . If M and N are two lattices of E , we define the generalised Sinnott index as follows

$$(M : N) = \begin{cases} |\det(\gamma)| & \text{if } \mathbb{F} = \mathbb{Q} \text{ or } \mathbb{R} \\ p^{v_p(\det(\gamma))} & \text{if } \mathbb{F} = \mathbb{Q}_p \end{cases}$$

where γ is an automorphism of the \mathbb{F} -vector space E such that $\gamma(M) = N$. Under this definition we have the following properties

Lemma 3.1. *Let L , M and N be three \mathcal{O} -submodules of a finite dimensional \mathbb{F} -vector space E . Then*

- (1) $(L : N) = (L : M)(M : N)$ whenever two of these symbols are defined.
- (2) If L is an \mathcal{O} -lattice of E , and if M is contained in L then M is an \mathcal{O} -lattice of E . The index $(L : M)$ is then defined precisely if $[L : M]$ is finite, in which case they coincide.

Proof. See for example [7, Lemma 1.1]. □

Definition 3.2. *The Sinnott module $U(K)$ is the $\mathbb{Q}[G]$ -module generated by the elements*

$$(\sum_{\sigma \in \text{Gal}(K/F)} \sigma)^{2r} \prod_{\mathfrak{p} | \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}})$$

where F browses the set of intermediate extensions in K/k .

Remark 3.3. *The Sinnott module is used along the generalised Sinnott index to calculate indices of modules of special units. The interested reader can refer to [6] or [7], for example, where this technique has been followed in the case of circular units or to [4] for the elliptic units case.*

We need the following result

Proposition 3.4. *$U(K)^{[S,r]}$ is a lattice of the vector space $e_{S,r} \mathbb{R}[G]$.*

Proof. It is enough to show that the \mathbb{Z} -rank of $U(K)$ is equal to the number of idempotents in $\{e_{\chi}, r_S(\chi) = r\}$, which amounts to prove that $e_{\chi} U(K) \neq 0$ ($e_{\chi} U(K) := e_{\chi}(U(K) \otimes \mathbb{C})$) for each character χ in $\{\chi, r_S(\chi) = r\}$. We can see that

$$\prod_{\mathfrak{p} | \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} = \prod_{\mathfrak{p} | \mathfrak{f}_F} (1 - \hat{\chi}(\sigma_{\mathfrak{p}})) e_{\chi^{-1}}$$

Let χ be a character such that $r_S(\chi) = r$. Then χ is not trivial since $|S| > r + 1$. In that case choose F such that $\mathfrak{f}_F | \mathfrak{f}_{\chi}$ and $F \subseteq K^{\chi}$. Consider now a prime $\mathfrak{p} | \mathfrak{f}_F$, \mathfrak{p} is ramified in F/k and so it is in K^{χ}/k and therefore $\hat{\chi}(\sigma_{\mathfrak{p}}) = 0$. So for this particular choice of F we have $\prod_{\mathfrak{p} | \mathfrak{f}_F} (1 - \sigma_{\mathfrak{p}}^{-1} e_{I_{\mathfrak{p}}}) e_{\chi^{-1}} = e_{\chi^{-1}}$ not trivial. We conclude that $U(K)$ has a \mathbb{Z} -rank equal to the number of idempotents in $\{e_{\chi}, r_S(\chi) = r\}$ and the result follows. □

Lemma 3.5. *The following generalized Sinnott indices are well defined*

- (1) $(U(K)^{[S,r]} : \omega_K U(K)^{[S,r]})$
- (2) $(\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]})$

$$(3) (\mathbb{R}[G]^{[S,r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]}))$$

Proof. The assertions (1) and (2) are a direct consequence of Proposition 3.4, Lemma 3.1 and the definition of the generalized Sinnott index. The index in (3) is defined since the image of $\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]}$ by the Rubin-Stark regulator is a lattice of $\mathbb{R}[G]^{[S,r]}$ as it is well known. \square

Corollary 3.6. *The generalized Sinnott index $(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]})$ is well defined and we have the equality*

$$(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}) = \frac{(\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]})}{(\mathbb{R}[G]^{[S,r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]}))} \cdot (U(K)^{[S,r]} : \omega_K U^{[S,r]}).$$

Proof. Lemmas 3.1 and 3.5 ensure that $(R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : R_w(\text{Stark}_K^{[S,r]}))$ is well defined. On the one hand, the map R_w is injective,

$$(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}) = (R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : R_w(\text{Stark}_K^{[S,r]})).$$

On the other hand, since $R_w(\text{Stark}_K^{[S,r]}) = \omega_K U(K)^{[S,r]}$ (see Proposition 2.3) holds, we obtain

$$\begin{aligned} & (R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : R_w(\text{Stark}_K^{[S,r]})) \\ &= \\ & \frac{(\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]})}{(\mathbb{R}[G]^{[S,r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]}))} (U(K)^{[S,r]} : \omega_K U(K)^{[S,r]}). \end{aligned}$$

Hence the corollary follows. \square

3.2. The class number Formula. Next, we use the previous result to prove the class number formula shown in Theorem 1.1.

Some further notations and definitions. Let F/k be an intermediate extension in K/k , we note by $\text{Ram}(F/k)$ the set of primes that ramify in the extension F/k . We assume in the present section that the set $\text{Ram}(K/k)$ is not empty. We make some further notations

- (1) We note by $A_{S,T}(F)$ (or simply $A_{S,T}$ when there is no risk of confusion) the S -ray class group of F modulo T .
- (2) $h_{S,T}$ will refer to the S -class number modulo T , i.e. $h_{S,T} := |A_{S,T}|$.
- (3) $Y_S(F) := \bigoplus_{w \in S_F} \mathbb{Z}w$ the free abelian group on S_F .
- (4) $X_S(F) := \{\sum a_w w \in Y_S(F), \sum a_w = 0\}$.
- (5) $\lambda_{S,T}(F) : U_{S,T}(F) \longrightarrow X_S(F) \otimes \mathbb{R}$ is the map defined by

$$\lambda_{S,T}(F)(\alpha) = -\sum_{w \in S_F} \log(|\alpha|_w) w.$$

- (6) $R_{S,T}(F) = |\det(\lambda_{S,T}(F))|$ the regulator associated to $\lambda_{S,T}(F)$.
- (7) μ_T the group of roots of unity in $U_{S,T}$ which is trivial in our case since we assumed that $U_{S,T}$ is torsion-free.
- (8) $\zeta_{S,T}$ The S -truncated zeta function defined on the half complex plane $\text{Re}(s) > 1$ by the Euler product

$$\zeta_{S,T} = \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s})^{-1} \prod_{\mathfrak{p} \in T} (1 - N\mathfrak{p}^{1-s}).$$

- (9) We assume that $\text{Ram}(K/k) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_{|\text{Ram}(K/k)|}\}$. Following the notation of Rubin in [5] we define for $I \subset (0, \dots, |\text{Ram}(K/k)|)$ the field

$$K_I := K^{<D_i, i \in I>}$$

where

$$\begin{cases} D_i \text{ is the decomposition subgroup of } \mathfrak{p}_i \text{ in } K/k \text{ if } i \neq 0 \\ D_i = \{1\} \text{ if } i = 0, \end{cases}$$

and $\langle D_i, i \in I \rangle$ is the subgroup of G generated by the decomposition groups $\{D_i, i \in I\}$.

- (10) For $I \subset (0, \dots, |\text{Ram}(K/k)|)$ the quantities $\zeta_{K_I, S, T}$, $h_{K_I, S, T}$, $R_{K_I, S, T}$ etc. are defined exactly in the same way as above but for the field K_I instead of the field K .

Proposition 3.7.

$$(\mathbb{R}[G]^{[S, r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S, T}(K)^{[S, r]})) = R_{K, S, T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} R_{K_I, S, T}^{(-1)^{|I|}}$$

Proof. Let $S = S_\infty \cup V$ and let $\mathcal{L}_{S, \infty}$ the map defined in (1). The facts that $e_{S, r} \mathbb{R}V_K = 0$ ($|S| > r + 1$) and the map

$$e_{S, r} \mathcal{L}_{S, \infty} : e_{S, r} \mathbb{R}U_{S, T}(K) \xrightarrow{e_{S, r} \mathcal{L}_S} e_{S, r} \mathbb{R}S_K \xrightarrow{\text{id}} e_{S, r} \mathbb{R}S_{\infty, K}$$

is an isomorphism, show that

$$(\mathbb{R}S_{\infty, K}^{[S, r]} : \mathcal{L}_{S, \infty}^{[S, r]}(\mathbb{R}U_{S, T}(K)^{[S, r]})) = \det(e_{S, r} \mathcal{L}_S).$$

Then, using the facts

$$\begin{aligned} (\mathbb{R}S_{\infty, K}^{[S, r]} : \mathcal{L}_{S, \infty}^{[S, r]}(\mathbb{R}U_{S, T}(K)^{[S, r]})) &= \left(\bigwedge_{\mathbb{R}[G]}^r \mathbb{R}S_{\infty, K}^{[S, r]} : \bigwedge_{\mathbb{R}[G]}^r \mathcal{L}_{S, \infty}^{[S, r]} \left(\bigwedge_{\mathbb{R}[G]}^r \mathbb{R}U_{S, T}(K)^{[S, r]} \right) \right) \\ &= \left(\mathbb{R}[G]^{[S, r]}(w_1 \wedge \dots \wedge w_r) : R_w \left(\bigwedge_{\mathbb{R}[G]}^r \mathbb{R}U_{S, T}(K)^{[S, r]} \right) (w_1 \wedge \dots \wedge w_r) \right) \\ &= \left(\mathbb{R}[G]^{[S, r]} : R_w \left(\bigwedge_{\mathbb{R}[G]}^r \mathbb{R}U_{S, T}(K)^{[S, r]} \right) \right) \\ &= \left(\mathbb{R}[G]^{[S, r]} : R_w \left(\bigwedge_{\mathbb{Z}[G]}^r U_{S, T}(K)^{[S, r]} \right) \right), \end{aligned}$$

we obtain $(\mathbb{R}[G]^{[S, r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S, T}(K)^{[S, r]})) = \det(e_{S, r} \mathcal{L}_S)$. Let v be a place of F , we denote $[w : v] := [K_w : F_v]$. Since the map

$$\begin{array}{ccc} \mathbb{R}X_S(F) & \longrightarrow & \mathbb{R}X_S(K) \\ v & \longmapsto & \frac{1}{|H|} \sum_{w|v} [w : v] w \end{array}$$

is an isomorphism for any $k \subset F \subset K$, it follows that $\det(e_\chi \lambda_{S, T}(K)) = \det(e_{\tilde{\chi}} \lambda_{S, T}(F))$, where $\tilde{\chi} \in \widehat{\text{Gal}(F/k)}$ and χ is a lift of $\tilde{\chi}$ to G . Therefore

$$\begin{aligned} R_{K_I, S, T} &= \prod_{\chi \in \widehat{\text{Gal}(K_I/k)}} \det(e_\chi \lambda_{S, T}(K_I)) \\ &= \prod_{\chi \in \widehat{G}, \chi(\text{Gal}(K/K_I)=1)} \det(e_\chi \lambda_{S, T}(K)). \end{aligned}$$

A simple inclusion-exclusion argument gives

$$\prod_{\chi, r_S(\chi)=r} \det(e_\chi \lambda_S(K)) = R_{K, S, T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} R_{K_I, S, T}^{(-1)^{|I|}}.$$

Using the fact that

$$\det(e_{S, r} \mathcal{L}_S) = \det(e_{S, r} \lambda_{S, T}(K)) = \prod_{\chi, r_S(\chi)=r} \det(e_\chi \lambda_{S, T}(K))$$

we see that

$$(\mathbb{R}[G]^{[S, r]} : R_w(\bigwedge_{\mathbb{Z}[G]}^r U_{S, T}(K)^{[S, r]})) = R_{K, S, T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} R_{K_I, S, T}^{(-1)^{|I|}}.$$

□

We prove now Theorem 1.1

Theorem 1.2. *The index $(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]})$ is finite, and we have*

$$[\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}] = [\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]}] h_{K,S,T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} h_{K_I, S, T}^{(-1)^{|I|}}.$$

Proof. We begin by the expression obtained in Corollary 3.6 and analyse each term. We have

$$(U(K)^{[S,r]} : \omega_K U(K)^{[S,r]}) = |\det(m_{\omega_K})|$$

where $\omega_K := \sum_{\chi \in \hat{G}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) e_{\chi^{-1}}$ and m_{ω_K} is the multiplication by ω_K . Since the set $\{e_{\chi}, r_S(\chi) = r\}$ is an \mathbb{R} -base of the vector space $e_{S,r} \mathbb{R}[G]$, and $U(K)^{[S,r]}$ is a lattice of it,

$$\det(m_{\omega_K}) = \prod_{\chi \in \hat{G}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi})$$

A simple inclusion-exclusion argument gives

$$\prod_{\chi \in \hat{G}, r_S(\chi)=r} L_{S,T}^{(r)}(0, \hat{\chi}) = \zeta_{K,S,T}^*(0) \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} \zeta_{K_I, S, T}^*(0)^{(-1)^{|I|}}$$

where $\zeta_{K_I, S, T}^*(0)$ is the first non trivial term in the Taylor expansion of the function $\zeta_{K_I, S, T}(s)$ at 0 given by

$$\zeta_{K_I, S, T}^*(0) := \lim_{s \rightarrow 0} s^{1-|S|} \zeta_{K_I, S, T}(s)$$

Recall the following well known S -class number formula (see e.g. [2])

$$\zeta_{K_I, S, T}^*(0) = -\frac{h_{K_I, S, T} R_{K_I, S, T}}{|\mu_T|}$$

This formula combined with the previous work gives

$$(U^{[S,r]} : \omega_K U^{[S,r]}) = h_{K, S, T} R_{K, S, T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} h_{K_I, S, T}^{(-1)^{|I|}} R_{K_I, S, T}^{(-1)^{|I|}}.$$

Using Proposition 3.7 and Corollary 3.6, we get

$$(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}) = (\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]}) h_{K, S, T} \prod_{I \subset (1, \dots, |\text{Ram}(K/k)|)} h_{K_I, S, T}^{(-1)^{|I|}}.$$

The index $(\mathbb{R}[G]^{[S,r]} : U(K)^{[S,r]})$ is known to be finite (see e.g. [7])

hence $(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}) = [\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(K)^{[S,r]} : \text{Stark}_K^{[S,r]}]$ is finite and this gives the desired result. \square

4. THE CLASS NUMBER FORMULA AND THE CYCLOTOMIC TOWER

In this section, we investigate the behavior of the class number formula along the cyclotomic \mathbb{Z}_p -extension of the field K . In all this section $K_\infty = \bigcup_{n \geq 0} K_n$ is the cyclotomic \mathbb{Z}_p -extension of K . We begin by the following remark

Remark 4.1. *Since K is a totally real number field, all layers K_n are totally real and verify the hypotheses made in the introduction. Furthermore, since we assumed that $\text{Ram}(K/k)$ is not empty then so is $\text{Ram}(K_n/k)$ for all $n \geq 1$.*

We prove now Theorem 1.2.

Theorem 1.3. *There exists factors B_n bounded asymptotically for n sufficiently large such that*

$$[\bigwedge_{\mathbb{Z}[G_n]}^r U_{S,T}(K_n)^{[S,r]} : \text{Stark}_{K_n}^{[S,r]}] = h_{K_n, S, T} B_n.$$

Proof. Theorem 1.1 gives for $n \geq 0$ (where $K_0 := K$)

$$[\bigwedge_{\mathbb{Z}[G_n]}^r U_{S,T}(K_n)^{[S,r]} : \text{Stark}_{K_n}^{[S,r]}] = h_{K_n,S,T}[\mathbb{R}[G_i]^{[S,r]} : U(K_n)^{[S,r]}] \prod_{I \subset (1, \dots, |\text{Ram}(K_n/k)|)} h_{K_n,I,S,T}^{(-1)^{|I|}}$$

Let

$$B_n := [\mathbb{R}[G_n]^{[S,r]} : U(K_n)^{[S,r]}] \prod_{I \subset (1, \dots, |\text{Ram}(K_n/k)|)} h_{K_n,I,S,T}^{(-1)^{|I|}}$$

All we have to do is to show that, under this definition, B_n is bounded for n sufficiently large.

First, as no finite place is totally split in the cyclotomic tower the product $\prod_{I \subset (1, \dots, |\text{Ram}(K_n/k)|)} h_{K_n,I,S,T}^{(-1)^{|I|}}$ has a finite number of factors which does not depend on n when this latter is sufficiently large. For the exact same reason, the fields K_I are all finite, and hence the product $\prod_{I \subset (1, \dots, |\text{Ram}(K_n/k)|)} h_{K_n,I,S,T}^{(-1)^{|I|}}$ is bounded independently of n when this latter is large enough. Furthermore, the Sinnott indices $[\mathbb{R}[G_n]^{[S,r]} : U(K_n)^{[S,r]}]$ are well known to be bounded along the cyclotomic tower (see e.g. [6], [7] or [4]) and this ends our proof. \square

REFERENCES

- [1] **Burns, B., Greither, C.** *On the Equivariant Tamagawa Number Conjecture for Tate motives*, Inventiones math. 153 (2003), 303-359.
- [2] **Gross, B. H.** *On values of abelian L-functions at $s = 0$* . J. Fac. Sci. Univ. Tokyo. 35, (1988), 177-197.
- [3] **Mazigh, Y.** *Iwasawa theory of Rubin-Stark units and class groups*. manuscripta math. (2016) DOI: 10.1007/s00229-016-0889-0.
- [4] **Oukhaba, H.** *Indice des unités elliptiques dans les \mathbb{Z}_p -extensions*. Bull. Soc. Math. France. 135 (2), (2007), 299-322.
- [5] **Rubin, K.** *Stark units and Kolyvagin's "Euler systems"*. J. reine angew. Math. 425, (1992), 141-154.
- [6] **Sinnott, W.** *On the Stickelberger ideal and the circular units of a cyclotomic field*. Ann. of Math. (2), 108(1), (1978), 107-134.
- [7] **Sinnott, W.** *On the Stickelberger ideal and the circular units of an abelian field*. Invent. Math. 62(2), (1980/81), 181-234.
- [8] **Rubin, K.** *A Stark conjecture "over \mathbb{Z} " for abelian L-functions with multiple zeros*. Ann. Inst. Fourier (Grenoble), 46(1):33-62, 1996.
- [9] **Tate, J.** *Les conjectures de Stark sur les fonctions L d'Artin en $s=0$* . Birkhäuser Boston Inc, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

⁽¹⁾ UNIVERSITÉ MOULAY ISMAÏL, DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE MEKNÈS, B.P. 11201 ZITOUNE, MEKNÈS, MAROC.

⁽²⁾ UNIVERSITÉ FRANCHE-COMTÉ, LABORATOIRE DE MATHÉMATIQUE, 16 ROUTE DE GRAY, 25030 BESANÇON, CEDEX, FRANCE.

E-mail address: youness.mazigh@univ-fcomte.fr